

PLANAR GRAPHS WITHOUT 5-CYCLES AND INTERSECTING TRIANGLES ARE (1, 1, 0)-COLORABLE

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ABSTRACT. A (c_1, c_2, \dots, c_k) -coloring of G is a mapping $\varphi : V(G) \mapsto \{1, 2, \dots, k\}$ such that for every $i, 1 \leq i \leq k$, $G[V_i]$ has maximum degree at most c_i , where $G[V_i]$ denotes the subgraph induced by the vertices colored i . Borodin and Raspaud conjecture that every planar graph without 5-cycles and intersecting triangles is $(0, 0, 0)$ -colorable. We prove in this paper that such graphs are $(1, 1, 0)$ -colorable.

1. INTRODUCTION

Graph coloring is one of the central topics in graph theory. A graph is (c_1, c_2, \dots, c_k) -colorable if the vertex set can be partitioned into k sets V_1, V_2, \dots, V_k , such that for every $i : 1 \leq i \leq k$ the subgraph $G[V_i]$ has maximum degree at most c_i . Thus a $(0, 0, 0)$ -colorable graph is properly 3-colorable.

The problem of deciding whether a planar graph is properly 3-colorable is NP-complete. A lot of research has been devoted to finding conditions for a planar graph to be properly 3-colorable. The well-known Grötzsch Theorem [8] shows that “triangle-free” suffices. The famous Steinberg Conjecture [13] proposes that “free of 4-cycles and 5-cycles” is also enough.

Conjecture 1.1 (Steinberg, [13]). *All planar graphs without 4-cycles and 5-cycles are 3-colorable.*

Some relaxations of the Steinberg Conjecture are known to be true. Along the direction suggested by Erdős to find a constant c such that a planar graph without cycles of length from 4 to c is 3-colorable, Borodin, Glebov, Raspaud, and Salavatipour [4] showed that $c \leq 7$, and more results similar to those can be found in the survey by Borodin [1]. Another direction of relaxation of the conjecture is to allow some defects in the color classes. Chang, Havet, Montassier, and Raspaud [6] proved that all planar graphs without 4-cycles or 5-cycles are $(2, 1, 0)$ -colorable and $(4, 0, 0)$ -colorable. In [10, 11, 16], it is shown that planar graphs without 4-cycles or 5-cycles are $(3, 0, 0)$ - and $(1, 1, 0)$ -colorable. Some more results along this directions can be found in the papers by Wang *et al.* [16, 17].

Havel [9] proposed that planar graphs with triangles far apart should be properly 3-colorable, which was confirmed in a recent preprint of Dvořák, Král and Thomas [7]. Borodin and Raspaud [5] combined the ideas of Havel and Steinberg and proposed the following so called Bordeaux Conjecture in 2003.

Conjecture 1.2 (Borodin and Raspaud, [5]). *Every planar graph without intersecting triangles and without 5-cycles is 3-colorable.*

A planar graph without intersecting triangles means the distance between triangles is at least 1. Let d^∇ denote the smallest distance between any pair of triangles in a planar graph. A relaxation of the Bordeaux Conjecture with $d^\nabla \geq 4$ was confirmed by Borodin and Raspaud [5], and the result was improved to $d^\nabla \geq 3$ by Borodin and Glebov [2] and, independently, by Xu [14]. Borodin and Glebov [3] further improved the result to $d^\nabla \geq 2$.

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Using the relaxed coloring notation, Xu [15] proved that all planar graphs without adjacent triangles and 5-cycles are $(1, 1, 1)$ -colorable, where two triangles are adjacent if they share an edge.

Let \mathcal{G} be the family of plane graphs with $d^\nabla \geq 1$ and without 5-cycles. Yang and Yerger [18] showed that planar graphs in \mathcal{G} are $(4, 0, 0)$ - and $(2, 1, 0)$ -colorable, but there is a flaw in one of their key lemmas (Lemma 2.4). In [12], we showed that graphs in \mathcal{G} are $(2, 0, 0)$ -colorable.

In this paper, we will prove another relaxation of the Bordeaux Conjecture. Let G be a graph and H be a subgraph of G . We call (G, H) to be *superextendable* if each $(1, 1, 0)$ -coloring of H can be extended to G so that vertices in $G - H$ have different colors from their neighbors in H ; in this case, we call H to be a superextendable subgraph.

Theorem 1.3. *Every triangle or 7-cycle of a planar graph in \mathcal{G} is superextendable.*

As a corollary, we have the following relaxation of the Bordeaux Conjecture.

Theorem 1.4. *A planar graph in \mathcal{G} is $(1, 1, 0)$ -colorable.*

To see the truth of Theorem 1.4 by way of Theorem 1.3, we may assume that the planar graph contains a triangle C since G is $(0, 0, 0)$ -colorable if G has no triangle. Then color the triangle, and by Theorem 1.3, the coloring of C can be superextended to G . Thus, we get a coloring of G .

As many results with similar fashion, we use a discharging argument to prove Theorem 1.3. This argument consists of two parts: structures and discharging. After introduce some common notations in Section 2, we show in Section 3 some useful special structures in a minimal counterexample to the theorem, then in Section 4, we design a discharging process to distribute the charges and use the special structures to reach a contradiction.

It should be noted that while the proof of our main theorem shares a lot of common properties with the $(2, 0, 0)$ result in [12], it is much more involved. We have to extend some powerful tools from [15] by Xu, and discuss in detail the structures around 4-vertices and 5-vertices. It would be interesting to know how to use the new tools developed in this paper to improve our result.

2. PRELIMINARIES

In this section, we introduce some notations used in the paper.

Graphs mentioned in this paper are all simple. For a positive integer n , let $[n] = \{1, 2, \dots, n\}$. A k -vertex (k^+ -vertex, k^- -vertex) is a vertex of degree k (at least k , at most k). The same notation will apply to faces and cycles. We use $b(f)$ to denote the vertex sets on f . We use $F(G)$ to denote the set of faces in G . An (l_1, l_2, \dots, l_k) -face is a k -face $v_1 v_2 \dots v_k$ with $d(v_i) = l_i$, respectively. A face f is a *pendant 3-face* of vertex v if v is not on f but is adjacent to some 3-vertex on f . A *pendant neighbor* of a 3-vertex v on a 3-face is the neighbor of v not on the 3-face.

Let C be a cycle of a plane graph G . We use $int(C)$ and $ext(C)$ to denote the sets of vertices located inside and outside C , respectively. The cycle C is called a *separating cycle* if $int(C) \neq \emptyset \neq ext(C)$, and is called a *nonseparating cycle* otherwise. We still use C to denote the set of vertices of C .

Let S_1, S_2, \dots, S_l be pairwise disjoint subsets of $V(G)$. We use $G[S_1, S_2, \dots, S_l]$ to denote the graph obtained from G by identifying all the vertices in S_i to a single vertex for each $i \in \{1, 2, \dots, l\}$.

A vertex v is *properly colored* if all neighbors of v have different colors from v . A vertex v is *nicely colored* if it shares a color (say i) with at most $\max\{s_i - 1, 0\}$ neighbors, where s_i is the deficiency allowed for color i ; thus if a vertex v is nicely colored by a color i which allows deficiency $s_i > 0$, then an uncolored neighbor of v can be colored by i .

3. SPECIAL CONFIGURATIONS

Let (G, C_0) be a minimum counterexample to Theorem 1.3 with minimum $\sigma(G) = |V(G)| + |E(G)|$, where C_0 is a triangle or a 7-cycle in G that is precolored. For simplicity, let $F_k = \{f : f \text{ is a } k\text{-face and } b(f) \cap C_0 = \emptyset\}$, $F'_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 1\}$, and $F''_k = \{f : f \text{ is a } k\text{-face and } |b(f) \cap C_0| = 2\}$.

The following lemmas are shown in [12].

Proposition 3.1 (Prop 3.1 in [12]). (a) *Every vertex not on C_0 has degree at least 3.*

(b) *A k -vertex in G can have at most one incident 3-face.*

(c) *No 3-face and 4-face in G can have a common edge.*

Lemma 3.2 (Lemma 3.2 in [12]). *The graph G contains neither separating triangles nor separating 7-cycles.*

Lemma 3.3 (Lemma 3.3 in [12]). *If G has a separating 4-cycle $C_1 = v_1v_2v_3v_4v_1$, then $\text{ext}(C_1) = \{b, c\}$ such that v_1bc is a 3-cycle. Furthermore, the 4-cycle is the unique separating 4-cycle.*

Lemma 3.4 (Lemma 3.4 in [12]). *If $x, y \in C_0$ with $xy \notin E(C_0)$, then $xy \notin E(G)$ and $N(x) \cap N(y) \subseteq C_0$.*

Lemma 3.5 (Lemma 3.6 in [12]). *Let u, w be a pair of diagonal vertices on a 4-face. If at most one of u and w is incident to a triangle, $G[\{u, w\}] \in \mathcal{G}$.*

Lemma 3.6 (Lemma 3.7 in [12]). *Let f be a face in $F_4 \cup F'_4$. Then*

- (1) *if $b(f) \cap C_0 = \{u\}$, then each of u and w is incident to a triangle.*
- (2) *if $f = uvwx \in F_4$ is a face with $d(u) = d(w) = 3$, then each of v and x is incident to a triangle.*

By Lemma 3.2, we may assume that C_0 is the boundary of the outer face of G .

Lemma 3.7. *In $\text{int}(C_0)$, let v and u be two adjacent 3-vertices. Then each vertex in $(N(u) \cup N(v)) \setminus \{u, v\}$ has degree at least 4.*

Proof. Suppose to the contrary that v_1 is a neighbor of v that has degree 3. Let $G' = G - \{u, v\}$. By the minimality of G , (G', C_0) is superextendable. Recolor v_1 properly, and then color u properly. Now v can be colored, or the three neighbors of v are colored differently. In the latter case, 1 or 2 (say 1) is used on u or v_1 . Then we color v with 1, a contradiction. \square

Lemma 3.8. *Let $f = uvw$ be a face in F_3 . Then each of the following holds.*

- (1) *If $d(u) = d(v) = 3$, then $d(w) \geq 5$.*
 - (2) *If f is a $(3, 3, 5^+)$ -face, then each pendant neighbor of u or v is either on C_0 or has degree at least 4.*
 - (3) *If f is a $(3, 4, 4)$ -face, then the pendant neighbor of u is either on C_0 or has degree at least 4 and at least one of the neighbors (not on f) of each 4-vertex is either on C_0 or has degree at least 4.*
- Consequently, a 4-vertex cannot be incident to a $(3, 4, 4)$ -face and a $(3, 4, 3, 4^+)$ -face from F_4 .*

Proof. (1) Suppose otherwise that $f = uvw$ is a $(3, 3, 4^-)$ -face. Let $G' = G - \{u, v\}$. It follows that $\sigma(G') < \sigma(G)$. By the minimality of G , (G', C_0) is superextendable. Recolor w properly and then color u properly. Then v can be colored, or $N(v)$ contains three different colors. In the latter case, 1 or 2 (say 1) is used on u or w , then we can color v with 1, a contradiction.

(2) Let $f = uvw$ be a $(3, 3, 5^+)$ -face. Let u' be the pendant neighbor of u . Assume that u' is not on C_0 . Suppose otherwise that $d(u') = 3$. By Lemma 3.7, each vertex in $(N(u) \cup N(v)) \setminus \{u, v\}$ has degree at least 4. So $d(u') \geq 4$, a contradiction.

(3) Let $f = uvw$ be a $(3, 4, 4)$ -face. Let u' be the pendant neighbor of u . Assume that u' is not on C_0 . Suppose otherwise that $d(u') = 3$. By the minimality of G , $(G - \{u, u'\}, C_0)$ is superextendable. Color u' properly. Then u can be colored, or $N(u)$ contains three different colors. In the latter case, if u' is colored

with 1 or 2, then we color u with the color of u' . Thus, we may assume that u' is colored with 3, and assume that v, w are colored with 1, 2 respectively and neither is nicely colored. If the neighbors of v not on f are colored with 1 and 2, then we recolor v with 3 and color u with 1. So, we may assume that they are colored with 1 and 3. Similarly, we may assume that the neighbors of w not on f are colored with 2 and 3. Now we switch the color of v and w , and color u with 1, a contradiction.

Now let v_1, v_2 be the two neighbors of v not on f . Suppose otherwise that $d(v_1) = d(v_2) = 3$ and $v_1, v_2 \notin V(C_0)$. By the minimality of G , $(G - \{u, v, w, v_1, v_2\}, C_0)$ is superextendable. We properly color v_1, v_2, w and u in order. Then v can be properly colored, or $N(v)$ has three different colors. In the latter case, only one vertex in $\{u, w, v_1, v_2\}$ is colored with 1 or 2 (say 1), so we color v with 1, a contradiction. \square

Lemma 3.9. *Let v be a k -vertex with $N(v) = \{v_i : i \in [k]\}$. Then each of the following holds.*

- (1) *For $k = 4$, if v is incident to a $(3, 4, 4)$ -face $f_1 = v_1 v v_2$ from F_3 and a 4-face $f_2 = v v_3 v_4$ from F_4 with $d(u) = 3$, then both v_3 and v_4 are incident to triangles. Consequently, f_2 cannot be a $(3, 3, 4, 4^+)$ -face.*
- (2) *For $k = 5$, let v be incident to a $(3, 4^-, 5)$ -face $f_1 = v_1 v_2 v$ from F_3 and two 4-faces $f_2 = v v_3 v_4$ and $f_3 = v v_4 v_5$ from F_4 . If $d(u) = d(w) = 3$, then at least two vertices in $\{v_3, v_4, v_5\}$ are incident to triangles.*

Proof. (1) Suppose otherwise that at most one vertex in $\{v_3, v_4\}$ is incident to a triangle. Let $G' = G[\{v_3, v_4\}]$ and v' be the new vertex. By Lemma 3.5, $G' \in \mathcal{G}$. Then $(G' - \{v, v_1, v_2, u\}, C_0)$ is superextendable. We color v_3 and v_4 with the color of v' , then properly color v_2, v_1, u in order. Then v can be properly colored, or $N(v)$ has three different colors. In the latter case, 1 or 2 (say 1) is used on v_1 or v_2 , so we color v with 1, a contradiction.

(2) Suppose otherwise that at most one vertex in $\{v_3, v_4, v_5\}$ is incident to a triangle. Let $G' = G[\{v_3, v_4, v_5\}]$, and let v' be the new vertex. By lemma 3.5, $G' \in \mathcal{G}$. Then $(G' - \{v, u, w\}, C_0)$ is superextendable. Color v_3, v_4, v_5 with the color on v' , and then properly color u and w since $d(u) = d(w) = 3$. We uncolor v_1, v_2 and then recolor v_2, v_1 properly in the order. Then v can be properly colored, or $N(v)$ has three different colors. In the latter case, 1 or 2 (say 1) is used on v_1 or v_2 , so we can color v with 1, a contradiction. \square

We first prove the following useful lemma.

Lemma 3.10. *Let v be a 4-vertex in $\text{int}(C_0)$ with $N(v) = \{v_i : i \in [4]\}$. If v is incident to two 4-faces that share an edge, then there is no t -path from v_i to v_{i+2} with $t \in \{1, 2, 3, 5\}$, where the subscripts of v are taken modulo 4.*

Proof. As v is incident to two 4-faces that share an edge, in any embedding, v_i and v_{i+2} cannot be in the same face, for otherwise, they will be in a separating 4-cycle, contrary to Lemma 3.3. Suppose otherwise that P is a t -path from v_i to v_{i+2} with $t \in \{1, 2, 3, 5\}$. Consider cycle $C = v_i P v_{i+2} v v_i$. If $t = 1$ or 5, then C is a 3- or 7-cycle separating v_{i+1} and v_{i+3} , a contradiction to Lemma 3.2; if $t = 2$, then C is a 4-cycle separating v_{i+1} and v_{i+3} , a contradiction to Lemma 3.3; if $t = 3$, then C is a 5-cycle, a contradiction to $G \in \mathcal{G}$. \square

Let v be a 4-vertex with its neighbor v_1, v_2, v_3, v_4 in the clockwise order in the embedding. Then v is called (v_i, v_{i+2}) -behaved if at most one of v_i and v_{i+2} is incident to a triangle.

Lemma 3.11. *Let v be a 4-vertex in $\text{int}(C_0)$ with $N(v) = \{v_i : i \in [4]\}$. Then each of the following holds.*

- (1) *If v is incident to two 4-faces $f_i = v v_i u v_{i+1}$ and $f_{i+1} = v v_{i+1} u_{i+1} v_{i+2}$ with $f_i, f_{i+1} \in F_4$, and at most one of $\{v_i, v_{i+1}, v_{i+2}\}$ is incident to a triangle, then $d(u_i) \geq 4$ or $d(u_{i+1}) \geq 4$, where the subscripts of u and v are taken modulo 4.*

- (2) If v is incident to two 4-faces $f_i = vv_iu_i v_{i+1}$ and $f_{i+2} = vv_{i+2}u_{i+2}v_{i+3}$ with $f_i, f_{i+2} \in F_4$, and at most one vertex from each of $\{v_i, v_{i+1}\}$ and $\{v_{i+2}, v_{i+3}\}$ is incident to a triangle, then $d(u_i) \geq 4$ or $d(u_{i+2}) \geq 4$, where the subscripts of u and v are taken modulo 4.
- (3) The vertex v is incident to at most one $(3, 3, 4, 4^+)$ -face from F_4 .
- (4) Let v be incident to two 4-faces that share an edge. If v is (v_1, v_3) -behaved and (v_2, v_4) -behaved, then none of the 4-faces can be $(3, 3, 4, 4^+)$ -face.

Proof. (1) By symmetry we assume that $i = 1$. Suppose otherwise that $d(u_1) = d(u_2) = 3$. Let $G' = G[\{v_1, v_2, v_3\}]$. Since at most one vertex in $\{v_1, v_2, v_3\}$ is incident to a triangle, by Lemma 3.5, $G' \in \mathcal{G}$. Thus, (G', C_0) is superextendable. Color v_1, v_2 and v_3 with the color of the resulting vertex of identification and then we can recolor u_1, u_2 and v properly, a contradiction.

(2) By symmetry we assume that $i = 1$. Suppose otherwise that $d(u_1) = d(u_3) = 3$. Let $G' = G[\{v_1, v_2\}, \{v_3, v_4\}]$. Let v' and v'' be the new vertices by identifying v_1 with v_2 , and v_3 with v_4 , respectively. Since at most one vertex from each of $\{v_1, v_2\}$ and $\{v_3, v_4\}$ is incident to a triangle, by Lemma 3.5, $G' \in \mathcal{G}$. Thus (G', C_0) is superextendable. Color v_1, v_2 with the color of v' and color v_3, v_4 with the color of v'' , then we can recolor v, u_1 and u_3 properly, a contradiction.

(3) Suppose otherwise that v is incident to at least two $(3, 3, 4, 4^+)$ -faces $f_1, f_2 \in F_4$. If f_1 and f_2 share an edge, let $f_1 = vv_1u_1v_2$ and $f_2 = vv_2u_2v_3$, then $d(u_1) = d(u_2) = 3$. We first show that $d(v_2) \geq 4$. Assume that $d(v_2) = 3$. Since u_1 and v_2 are two adjacent 3-vertices in $\text{int}(C_0)$, so by Lemma 3.7, $(N(u_1) \cup N(v_2)) \setminus \{u_1, v_2\}$ has degree at least 4, which implies that $d(u_2) \geq 4$, a contradiction. Thus f_1 is a $(3, 3, 4, 4^+)$ -face with $d(u_1) = d(v_1) = 3$ and f_2 is a $(3, 3, 4, 4^+)$ -face with $d(u_2) = d(v_3) = 3$. By Proposition 3.1(c), none of v_1 and v_3 is incident to a triangle. So by (1), $d(u_1) \geq 4$ or $d(u_2) \geq 4$, a contradiction to $d(u_1) = d(u_2) = 3$. If f_1 and f_2 do not share an edge, then it contradicts to (2).

(4) Assume that v is incident to a $(3, 3, 4, 4^+)$ -face f_1 . Then by symmetry $d(v_1) = d(u_1) = 3$ or $d(u_1) = d(v_2) = 3$. First we assume that $d(v_1) = d(u_1) = 3$. Let $G' = G - v$ and $H = G'[\{v_2, v_4\}]$. By Lemma 3.10, there is no t -path from v_2 to v_4 with $t \in \{1, 2, 3, 5\}$, so H contains no 5-cycle and no new triangles, in addition to the fact that G is (v_2, v_4) -behaved, H has no intersecting triangles, therefore $H \in \mathcal{G}$. Thus (H, C_0) is superextendable. Color v_2 and v_4 with the color of the new vertex, then v can be colored properly, or $N(v)$ has three different colors. Consider the latter case. Recolor u_1, v_1 properly in the order. If v_1 is colored with 1 or 2, then we color v with the color of v_1 ; if v_1 is colored 3, then color v with 3 and recolor v_1 with the color of u_1 . In either case, we reach a contradiction. Similar to the above argument, v cannot be incident to a $(3, 3, 4, 4^+)$ -face with $d(u_1) = d(v_2) = 3$. \square

For $k = 4, 5$, we call a k -vertex in $\text{int}(C_0)$ to be *poor* if it is incident to k 4-faces from F_4 . If a k -vertex is not poor, then we call it *rich*.

Lemma 3.12. *Let v be a poor 4-vertex with $N(v) = \{v_i : i \in [4]\}$ and four incident 4-faces $f_i = vv_iu_i v_{i+1}$ for $i \in [4]$, where the subscripts of v and u are taken modulo 5. Furthermore, v is (v_1, v_3) -behaved. If either $d(v_2) = 3$ or $d(v_2) = 4$ and v_2 is (u_1, u_2) -behaved, then $d(v_4) \geq 5$, or $d(v_4) = 4$ and v_4 is not (u_3, u_4) -behaved.*

Proof. Suppose to the contrary that $d(v_4) = 3$ or $d(v_4) = 4$ and v_4 is (u_3, u_4) -behaved.

Consider that $d(v_2) = d(v_4) = 3$. Let $G' = G - v$ and $H = G'[\{v_1, v_3\}]$. By Lemma 3.10, there is no t -path from v_1 to v_3 with $t \in \{1, 2, 3, 5\}$. It follows that H contains no 5-cycle and no new triangles. In addition to the fact that v is (v_1, v_3) -behaved, H has no intersecting triangles. Therefore, $H \in \mathcal{G}$. Thus (H, C_0) is superextendable. Color v_1 and v_3 with the color of the new vertex, and recolor v_2, v_4 properly. Then 1 or 2 (say 1) is used on v_2 or v_4 . Now color v with 1, a contradiction.

By symmetry, consider that $d(v_2) = 3$ and $d(v_4) = 4$. Let $G' = G - \{v, v_4\}$ and let $H = G'[\{v_1, v_3\}, \{u_3, u_4\}]$. Let v' and v'_4 be the new vertices by identifying v_1 with v_3 and u_3 with u_4 , respectively. As above, there is

no 5-cycle or new 3-cycle containing v' or v'_4 . Furthermore, if there is a 3-cycle, 5-cycle containing v' and v'_4 , then there is a 2-path or a 4-path from $\{v_1, v_3\}$ to $\{u_3, u_4\}$, thus there is 5-cycle or separating 7-cycle in G , a contradiction. Therefore, $H \in \mathcal{G}$. Note that now (H, C_0) is superextendable. Color v_1, v_3 with the color of v' and color u_3, u_4 with the color of v'_4 , then properly color v_2, v_4 . Now v can be colored, or $N(v)$ contains three different colors. In the latter case, 1 or 2 (say 1) is used on v_2 or v_4 , then color v with 1, a contradiction.

Consider $d(v_2) = d(v_4) = 4$. Let $G' = G - \{v, v_2, v_4\}$, and let $H = G'[\{u_1, u_2\}, \{v_1, v_3\}, \{u_3, u_4\}]$. Let v'_2, v', v'_4 be the new vertices by identifying u_1 with u_2 , v_1 with v_3 and u_3 with u_4 , respectively. As shown above, there is no 3-cycle or 5-cycle containing one of v'_2, v', v'_4 , or the pairs in $\{v'_2, v'\}, \{v', v'_4\}, \{v'_2, v'_4\}$. If there is a 3-cycle or 5-cycle containing v'_2, v' and v'_4 then there is 1- or 3-path from v'_2 to v'_4 or a 2-path from v' to v'_4 , but in either case, there is a 5-cycle or a separating 7-cycle, a contradiction. Thus, (H, C_0) is superextendable. Color the vertices with the color of their resulting vertex, respectively, then color v_2, v_4 properly. Now v can be colored, or $N(v)$ contains three different colors. In the latter case, 1 or 2 (say 1) is used on v_2 or v_4 , then we color v with 1, a contradiction. \square

Lemma 3.13. *Let v be a poor 5-vertex with $N(v) = \{v_i : i \in [5]\}$ and five incident 4-faces $f_i = vv_iu_iv_{i+1}$ for $i \in [5]$, where the subscripts of u and v are taken modulo 5. Suppose that at most one vertex in $N(v)$ is incident with a triangle. Then each of the following holds.*

- (1) *If $d(u_i) = d(v_i) = 3$ for some $i \in [5]$, then $d(u_j) \geq 4$ for $j \in [5] - \{i\}$.*
- (2) *At most two vertices in $\{u_i : i \in [5]\}$ have degree 3.*
- (3) *Let $d(u_i) = 3$. If v_j has degree 3 or is a 4-vertex with (u_{j-1}, u_j) -behaved, then $d(v_k) \geq 5$, or $d(v_k) = 4$ and v_k is not (u_{k-1}, u_k) -behaved, where $\{j, k\} = \{i - 1, i + 2\}$.*

Proof. (1) Without loss of generality, We may assume that $i = 1$. By Lemma 3.7, $d(u_5) \geq 4$. Suppose otherwise that $d(u_j) = 3$ for some $j \neq 1, 5$. Let $H = G'[\{v_j, v_{j+1}, v_{j+3}\}]$, where $G' = G - v$. By Lemma 3.5 and 3.12, $H \in \mathcal{G}$. So (H, C_0) is superextendable. In G' , color v_j, v_{j+1}, v_{j+3} with the color of the resulting vertex, and uncolor u_j, u_1, v_1 and recolor them properly in the order, we get a desired coloring of G' . Now v can be properly colored, or $N(v)$ contains three different colors. In the latter case, if v_1 is colored with 1 or 2, then color v with the color of v_1 ; if v_1 is colored with 3, then color v with 3 and recolor v_1 with the color of u_1 , a contradiction.

(2) Suppose otherwise that at least three vertices in $\{u_i : i \in [5]\}$ have degree 3. By symmetry, u_i, u_{i+1}, u_{i+2} have degree 3 or u_i, u_{i+1}, u_{i+3} have degree 3 for some $i \in [5]$. We may assume that $i = 1$. Let $d(u_1) = d(u_2) = d(u_3) = 3$. Consider $H = G[\{v_1, v_2, v_3, v_4\}]$. By Lemma 3.5, $H \in \mathcal{G}$. So (H, C_0) is superextendable. In G , color v_1, v_2, v_3, v_4 with the color of the resulting vertex and recolor u_1, u_2, u_3 properly and finally color v properly, a contradiction. Let $d(u_1) = d(u_2) = d(u_4) = 3$. Consider $H = G[\{v_1, v_2, v_3\}, \{v_4, v_5\}]$. Let v' and v'' be the resulting vertices by identifying v_1, v_2, v_3 and v_4, v_5 , respectively. By Lemma 3.5, $H \in \mathcal{G}$. So (H, C_0) is superextendable. In G , color v_1, v_2, v_3 with the color of v' and color v_4, v_5 with the color of v'' and recolor u_1, u_2, u_4 properly, and now v can be properly colored, a contradiction.

(3) Without loss of generality, We assume that $i = 2$ and $d(u_2) = 3$. Let $H = G[\{v_2, v_3\}] - u_2$ and the resulting vertex be v' . By symmetry, let $j = i - 1 = 1$ and $k = i + 2 = 4$. Suppose to the contrary that $d(v_4) = 3$ or $d(v_4) = 4$ and v_4 is (u_3, u_4) -behaved. By the proof of Lemma 3.12, we can get a desired coloring of H and the color of v is different from the color of v' . Then we color v_2 and v_3 with the color of v' and color u_2 properly, a contradiction. \square

4. DISCHARGING PROCEDURE

In this section, we will finish the proof of the main theorem by a discharging argument. Let the initial charge of vertex $v \in G$ be $\mu(v) = 2d(v) - 6$, and the initial charge of face $f \neq C_0$ be $\mu(f) = d(f) - 6$ and

$\mu(C_0) = d(C_0) + 6$. Then

$$\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = 0.$$

We will use the following special 4-faces from F_4 in the discharging.

- A $(3, 4, 4, 5)$ -face is *special* if none of the 4-vertices is incident to triangles.
- A $(3, 4, 4, 5)$ -face is *weak* if exactly one of the 4-vertices is incident to a triangle.
- A $(3, 4, 5, 5)$ - or $(3, 5, 4, 5)$ - or $(3, 5, 5, 5)$ -face is *special* if the 5-vertices on the face are poor.
- A $(4, 4, 4, 5)$ -face is *special* if the 5-vertex and the 4-vertices adjacent to the 5-vertex are poor.
- A $(4, 4, 5, 5)$ -face is *special* if the 4-vertices and 5-vertices are poor.
- A 4-face is *rich* if it contains two rich 5-vertices or 6^+ -vertices.

The discharging rules are as follows.

(R1) Let $v \notin C_0$. Then v gives charges in the following ways:

(R1.1) $d(v) = 4$

(R1.1.1) If v is rich, then v gives $\frac{5}{4}$ to each incident $(3, 4, 4)$ -face from F_3 and 1 to other 3-faces from F_3 , $\frac{1}{2}$ to each pendant 3-face from F_3 , 1 to each incident $(3, 3, 4, 4^+)$ -face from F_4 . Furthermore, if v is incident to a triangle, then v gives $\frac{3}{4}$ to its incident 4-face (other than $(3, 3, 4, 4^+)$ -face from F_4); if v is not incident to a triangle, then v distributes the remaining charges only to other incident 4-faces from F_4 evenly.

(R1.1.2) If v is poor, then v gives $\max\{0, \frac{2-w(f)}{|Q|}\}$ to f , where v is on 4-face f and Q is the set of poor 4-vertices on f , and $w(f)$ is the weight that f receives from vertices not in Q .

(R1.2) $d(v) = 5$

(R1.2.1) If v is rich, then v gives 2 to each incident $(3, 4^-, 5)$ -face from F_3 , and $\frac{3}{2}$ to other incident 3-faces from F_3 , $\frac{1}{2}$ to each pendant 3-face from F_3 . Furthermore, if v is incident to a triangle, then v gives 1 to its incident 4-face; if v is not incident to a triangle, then v distributes the remaining charges only to other incident 4-faces from F_4 evenly.

(R1.2.2) If v is poor, then v gives 1 to each incident $(3, 3, 5, 4^+)$ -face or $(3, 4, 5, 4)$ -face or special $(3, 4, 4, 5)$ -face, $\frac{3}{4}$ to each incident special $(3, 4, 5, 5)$ -, $(3, 5, 4, 5)$ -, $(3, 5, 5, 5)$ -, $(4, 4, 5, 5)$ -, $(4, 4, 4, 5)$ -face, or weak $(3, 4, 4, 5)$ -face, 0 to a rich 4-face, and $\frac{1}{2}$ to each other incident 4-face.

(R1.3) Each 6^+ -vertex gives 2 to each incident 3-face, $\frac{1}{2}$ to each pendant 3-face, and distributes the remaining charges to incident 4-faces evenly.

(R2) Each $v \in C_0$ gives $\frac{1}{2}$ to each pendant face from F_3 , 1 to each incident face from F_4'' , $\frac{3}{2}$ to each incident face from F_3'' or F_4' , and 3 to each incident face from F_3' .

(R3) C_0 gives 2 to each 2-vertex on C_0 , $\frac{3}{2}$ to each 3-vertex on C_0 , and 1 to each 4-vertex on C_0 . In addition, if C_0 is a 7-face with six 2-vertices, then it gains 1 from the incident face.

The following useful facts are from the rules.

Lemma 4.1. *The vertices and faces mentioned in this lemma are disjoint from C_0 .*

- (1) *If a 4-vertex is incident to a triangle, then it gives 1 to each incident $(3, 3, 4, 4^+)$ - or $(3, 4, 3, 4^+)$ -face, and at least $\frac{3}{4}$ to each other 4-face.*
- (2) *Each rich 4-vertex gives at least $\frac{1}{2}$ to each incident 4-face, and if it is not incident to $(3, 3, 4, 4^+)$ -face, then it gives at least $\frac{2}{3}$ to each incident 4-face.*
- (3) *Let $f = uvwx$ be a $(3, 4, 4, 4)$ -face with w not incident with a triangle. Then each rich 4-vertex on $b(f)$ gives at least $\frac{2}{3}$ to f .*
- (4) *A rich 5-vertex gives at least 1 to each incident 4-face. Moreover, if such a 5-vertex is incident to a triangle that is not a $(3, 4^-, 5)$ -face, then it gives at least $\frac{5}{4}$ to each incident 4-face. A 6^+ -vertex gives*

at least 1 to each incident 4-face. Moreover, if such a 6^+ -vertex is incident to a triangle, then it gives at least $\frac{4}{3}$ to each incident 4-face.

- (5) Let v be a poor 4-vertex on a 4-face f . Then v gives at most 1 to each incident $(3, 3, 4, 4^+)$ -face, at most $\frac{2}{3}$ to each incident $(3, 4, 4, 4)$ -face, at most $\frac{1}{4}$ to a $(4, 4^+, 4^+, 4^+)$ -face that is adjacent to a triangle and at most $\frac{1}{2}$ to each other incident 4-face.

Proof. (1) By (R1.1.1), we just need to show that when v is incident to a $(3, 4, 3, 4^+)$ or a $(3, 3, 4, 4^+)$ -face, v cannot be incident to a $(3, 4, 4)$ -face. But this is true by Lemma 3.8(3) and Lemma 3.9 (1).

(2) Let v be a rich 4-vertex, note that v is incident to at most three 4-faces. Suppose that v is incident to exactly one 4-face f . So if v is incident to a triangle, then by (R1.1.1), it gives at least $2 - \frac{5}{4} = \frac{3}{4}$ to f ; if v is not incident to a triangle but adjacent to pendant triangles, then it gives at least $2 - 2 \cdot \frac{1}{2} = 1$ to f ; otherwise, v gives at least 2 to f .

Let v be incident to exactly two 4-faces. Since G has no 5-cycle, v is not incident to a triangle. If v is not adjacent to a pendant triangle, then it gives at least 1 to each incident 4-face. Let v be adjacent to a pendant triangle. If v is not incident to $(3, 3, 4, 4^+)$ -face, then by (R1.1.1), v gives $\frac{2-\frac{1}{2}}{2} = \frac{3}{4}$ to each 4-face; if v is incident to a $(3, 3, 4, 4^+)$, then by Lemma 3.11(3), it is incident to exactly one $(3, 3, 4, 4^+)$ -face. By (R1.1.1), v gives $2 - \frac{1}{2} - 1 = \frac{1}{2}$ to the other 4-face.

If v is incident to exactly three 4-faces, by Lemma 3.11(3), it is incident to at most one $(3, 3, 4, 4^+)$ -face. If v is incident to a $(3, 3, 4, 4^+)$ -face, then by (R1.1.1), it gives at least $\frac{2-1}{2} = \frac{1}{2}$ to each incident 4-face, otherwise, v gives at least $\frac{2}{3}$ to each incident 4-face.

(3) By symmetry suppose that v or w is rich 4-vertices. By Lemma 3.11(1) and (4) v or w cannot be incident to a $(3, 3, 4, 4^+)$ -face that share an edge with f since w is not incident to a triangle. By (R1.1.1) v or w gives at least $\frac{2}{3}$ to f .

(4) Let v be a rich 5-vertex that is incident to $t_3 \leq 1$ triangles and s pendant 3-faces. Then v is incident to at most $(5 - 2t_3 - s - 1)$ 4-faces. By (R1.2.1), v gives at least $\frac{4-2t_3-\frac{1}{2}s}{5-2t_3-s-1} \geq 1$ to each incident 4-face. In particular, if v is incident to a triangle that is not a $(3, 4^-, 5)$ -face, then by (R1.2.1), v gives at least $\frac{4-\frac{3}{2}-\frac{1}{2}s}{5-2-s-1} \geq \frac{5}{4}$ to each incident 4-face.

Similarly, if v is a t -vertex with $t \geq 6$ that is incident to $t_3 \leq 1$ triangles and s pendant 3-faces, then v is incident to at most $(t - 2t_3 - s)$ 4-faces. By (R1.3), v gives at least $\frac{2t-6-2t_3-\frac{1}{2}s}{t-2t_3-s} = \frac{t-2t_3-\frac{1}{2}s+(t-6)}{t-2t_3-s} \geq 1$. Moreover, if $t_3 = 1$, then v is incident to at most $(t-s-3)$ 4-faces. In this case, v gives at least $\frac{(2t-6)-2-\frac{1}{2}s}{t-s-3} \geq \frac{2t-8}{t-3} \geq \frac{4}{3}$ to each incident 4-face.

(5) First assume that f is a $(3, 3, 4, 4^+)$ -face with $d(x) \geq 4$. If x is also a poor 4-vertex, then by (R1.1.2) both x and v give 1 to f . If x is not a poor 4-vertex, then by (R1.1.1), (R1.2.2) and (4), x gives at least 1 to f . In either case, by (R1.1.2) v gives at most 1 to f .

Second, assume that f is a $(3, 4, 4, 4)$ -face. Since v is poor, the 4-vertex not adjacent to 3-vertex on f is not incident to a triangle. By (3) and (R1.1.2), v gives at most $\frac{2}{3}$ to f .

Next, assume that $f = vuwx$ is a $(4, 4^+, 4^+, 4^+)$ -face that is adjacent to a triangle. Let w be incident to a triangle. If $d(w) = 4$, then both u and x are rich. By (2), (4) and (R1.2.2), u and x each gives at least $\frac{1}{2}$ to f and w gives at least $\frac{3}{4}$ to f . So by (R1.1.2) v gives at most $\frac{1}{4}$ to f . If $d(w) = 5$, then u or x is not poor. We assume, without loss of generality, that x is not poor. By (2)(4) and (R1.2.2), x gives at least $\frac{1}{2}$ to f . In this case, u and v may be both poor. It follows by (4) and (R1.1.2) that v gives at most $\frac{2-1-\frac{1}{2}}{2} = \frac{1}{4}$ to f . If $d(w) \geq 6$, then by (4) w gives at least $\frac{4}{3}$ to f . In this case, each of u, x and v may be poor. By (R1.1.2), v gives at most $\frac{2-\frac{4}{3}}{3} = \frac{2}{9} < \frac{1}{4}$ to f . Now by symmetry let u be incident to a triangle. Then $d(u) \geq 5$. It follows that either $d(u) \geq 6$ or w is not poor. In the former case, similarly, we can show that v gives at most $\frac{2}{9} < \frac{1}{4}$. In the latter case, by (2)(4) and (R1.2.2) w gives at least $\frac{1}{2}$ to f and u gives at least 1 to f . Note that x may be poor. Thus by (R1.1.2), v gives at most $\frac{2-1-\frac{1}{2}}{2} = \frac{1}{4}$ to f .

Finally, assume that f is a 4-face which is neither $(3, 3, 4, 4^+)$ nor $(3, 4, 4, 4)$ -face. By Lemma 3.6(2), the number of 3-vertices on f is at most two. Since f is not $(3, 3, 4, 4^+)$ -face, the number of 3-vertices on f is at most one. First consider that f contains no 3-vertex. If f is a rich 4-face, then by (4) each of the two rich 5-vertices or 6⁺-vertices gives at least 1 to f . In this case, by (R1.1.2) v gives 0 to f . If f is not a rich 4-face, then by (2) (4) and (R1.2.2), each of 4⁺-vertices on f not in Q gives at least $\frac{1}{2}$ to f , where Q is the set of poor 4-vertices on f . By (R1.1.2), v gives at most $\frac{2 - \frac{1}{2}(4 - |Q|)}{|Q|} = \frac{1}{2}$ to f .

Next consider that f contains one 3-vertex. Since f is not $(3, 4, 4, 4)$, it contains at least one 5⁺-vertex. On the other hand, since f contains one 3-vertex and one 4-vertex v , f contains at most two 5⁺-vertices. Assume first that f contains exactly two 5⁺-vertices. If both 5⁺-vertices are rich 5-vertices or 6⁺-vertices, by (4), each of them gives 1 to f . By (R1.1.2), v gives 0 to f . If exactly one of 5⁺-vertex is poor 5-vertex. Then by (4) and (R1.2.2), the poor 5-vertex gives at least $\frac{1}{2}$ to f and the other 5⁺-vertex gives at least 1 to f . Thus, by (R1.1.2) v gives at most $\frac{1}{2}$ to f . Thus, we may assume that both of the 5⁺-vertices must be poor 5-vertices. It follows that f is a special $(3, 4, 5, 5)$ or $(3, 5, 4, 5)$ -face. By (R1.1.2) and (R1.2.2), v gives at most $2 - 2 \cdot \frac{3}{4} = \frac{1}{2}$ to f .

Thus, assume that f contains one 5⁺-vertex. It follows that f is a $(3, 4, 4, 5^+)$ or $(3, 4, 5^+, 4)$ -face. If the 5⁺-vertex is not poor 5-vertex, then by (4), it gives at least 1 to f . If the other 4-vertex is rich, then by (2), it gives $\frac{1}{2}$ to f . Thus, by (R1.1.2), v gives at most $\frac{1}{2}$ to f . If the other 4-vertex is poor, then by (R1.1.2) again, v gives at most $\frac{1}{2}$ to f . Thus, we may assume that the 5⁺-vertex is a poor 5-vertex. In this case, f is a special $(3, 4, 4, 5)$ -face or weak $(3, 4, 4, 5)$ -face or $(3, 4, 5, 4)$ -face. By (R1.2.2), (R1.1.2), (1) and (2), v gives at most $\max\{\frac{2-1}{2}, 2 - 2 \cdot \frac{3}{4}\} = \frac{1}{2}$ to f . \square

Now we shall show that each $x \in V(G) \cup F(G)$ other than C_0 has final charge $\mu^*(x) \geq 0$ and $\mu^*(C_0) > 0$.

First we consider vertices in $\text{int}(C_0)$. Note that $\text{int}(C_0)$ contains no 2⁻-vertices by Proposition 3.1. As 3-vertices in $\text{int}(C_0)$ is not involved in the discharging process, they have final charge $2 \cdot 3 - 6 = 0$. By (R1.3), 6⁺-vertices have nonnegative final charges. Thus, we are left with 4-vertices and 5-vertices in $\text{int}(C_0)$.

In Lemmas 4.2 -4.5, when we discuss the case that v is a poor k -vertex for $k = 4, 5$, we assume that $N(v) = \{v_i : i \in [k]\}$ and $f_i = vv_iu_i v_{i+1}$ for $i \in [k]$ be the k incident 4-faces of v (the subscripts of u and v are taken modulo k). We further assume that v_1, v_2, \dots, v_k are in the clockwise order in the embedding.

Lemma 4.2. *Each 4-vertex $v \in \text{int}(C_0)$ has nonnegative final charge.*

Proof. First suppose that v is rich. Note that when v is incident with a 3-face, it is incident with at most one 4-face and at most one 3-face, since G has no 5-cycle and intersecting 3-cycle. By Lemma 3.11(3), v is incident to at most one $(3, 3, 4, 4^+)$ -face from F_4 . So by (R1.1.1), v gives out more than 2 only if v is incident to a $(3, 4, 4)$ -face from F_3 and a $(3, 3, 4, 4^+)$ -face from F_4 , which is impossible by Lemma 3.9 (1), or a $(3, 4, 4)$ -face from F_3 and two pendant 3-faces from F_3 , which is also impossible by Lemma 3.8. So v gives out at most 2, and its final charge is at least $2 \cdot 4 - 6 - 2 = 0$.

Next we assume that v is poor. We distinguish the following two cases.

Case 1. $N(v)$ has at least two vertices incident to triangles.

Assume that $N(v)$ has at least three vertices incident to triangles, without loss of generality, that each of v_1, v_2, v_3 is incident with a triangle. Since G contains no 5-cycle, $d(v_i) \geq 5$ for $i \in [3]$. By Lemma 4.1(4), v_i for $i \in [3]$ gives at least 1 to each incident 4-face. By (R1.1.2), v gives 0 to f_1 and f_2 , and at most 1 to f_3 and f_4 , respectively. Thus, $\mu^*(v) = 2 - 1 \cdot 2 = 0$. Thus, we assume that $N(v)$ has exactly two vertices incident with triangles.

First let the two vertices be v_1 and v_2 . By Lemma 4.1(4), f_1 gets at least 2 from v_1 and v_2 . By (R1.1.2), v gives 0 to f_1 . Since only each of v_1 and v_2 is incident with a 3-face, v is (v_1, v_3) -behaved and (v_2, v_4) -behaved.

By Lemma 3.11(4) none of f_i with $i \in [4]$ is a $(3, 3, 4, 4^+)$ -face. Thus v gives at most $\frac{2}{3}$ to each of f_2, f_3 and f_4 by Lemma 4.1(5). Thus, $\mu^*(v) \geq 2 - 3 \cdot \frac{2}{3} = 0$.

Then, by symmetry let the two vertices be v_1 and v_3 . Since G has no 5-cycle, $d(v_1) \geq 5$ and $d(v_3) \geq 5$. It follows that none of f_i for $i \in [4]$ is a $(3, 4, 4, 4)$ -face. If none of them is a $(3, 3, 4, 4^+)$ -face, then by Lemma 4.1(5), v gives at most $\frac{1}{2}$ to each f_i . Thus, $\mu^*(v) \geq 2 - 4 \cdot \frac{1}{2} = 0$. So we may assume that f_1 is a $(3, 3, 4, 4^+)$ -face, i.e., $d(u_1) = d(v_2) = 3$. By Lemma 3.7, $d(u_2) \geq 4$. By Lemma 3.11(1) and (2), $d(u_3), d(u_4) \geq 4$. This implies that only one of f_i , where $i \in [4]$, is a $(3, 3, 4, 4^+)$ -face.

Let $d(v_4) \geq 4$. By Lemma 4.1(5), v gives at most $\frac{1}{4}$ to each of f_3 and f_4 , at most 1 to f_1 and $\frac{1}{2}$ to f_2 . Thus, $\mu^*(v) \geq 2 - 1 - \frac{1}{2} - 2 \cdot \frac{1}{4} = 0$.

Let $d(v_4) = 3$. By Lemma 4.1(5), v gives at most $\frac{1}{2}$ to f_2 and f_3 , respectively. If $d(v_1) = 5$, then u_4 is rich since G is 5-cycle free. By Lemma 4.1(2), u_4 gives at least $\frac{1}{2}$ to f_4 . Note that the 5-vertex v_1 is incident to a 3-face and two 4-faces, $d(v_2) = d(v_4) = 3$ and at most one vertex in $\{u_1, v, u_4\}$ is incident with a triangle. By Lemma 3.9(2), the triangle incident with v_1 cannot be a $(3, 4^-, 5)$ -face. By Lemma 4.1(4), v_1 gives at least $\frac{5}{4}$ to each of f_1 and f_4 . Thus, v gives at most $2 - \frac{5}{4} - \frac{1}{2} = \frac{1}{4}$ to f_4 and $2 - \frac{5}{4} = \frac{3}{4}$ to f_1 . If $d(v_1) \geq 6$, then by Lemma 4.1(4), v_1 gives at least $\frac{4}{3}$ to each of f_1 and f_4 . Thus, v gives at most $2 - \frac{4}{3} = \frac{2}{3}$ to f_1 and at most $\frac{2 - \frac{4}{3}}{2} = \frac{1}{3}$ to f_4 . Therefore, $\mu^*(v) \geq 2 - 2 \cdot \frac{1}{2} - \max\{\frac{3}{4} + \frac{1}{4}, \frac{2}{3} + \frac{1}{3}\} = 0$.

Case 2. $N(v)$ has at most one vertex incident with a triangle.

In this case, v is (v_1, v_3) -behaved and (v_2, v_4) -behaved. It follows by Lemma 3.11(4) that no 4-faces incident to v is a $(3, 3, 4, 4^+)$ -face. On the other hand, if v is not incident to a $(3, 4, 4, 4)$ -face, then by Lemma 4.1(5), v gives at most $\frac{1}{2}$ to each incident 4-face. Thus $\mu^*(v) \geq 2 - 4 \cdot \frac{1}{2} = 0$. Therefore, we may assume that v is incident to a $(3, 4, 4, 4)$ -face, by symmetry, say f_1 such that $d(u_1) = 3$ or $d(v_2) = 3$.

Claim. We may assume that none of f_2, f_3, f_4 is a $(3, 4, 4, 4)$ -face.

Proof of Claim. We may assume that $d(v_2) = 3$. For otherwise, let $d(u_1) = 3$. Then by Lemma 3.11(1) and (2), $d(u_i) \geq 4$ for $i = 2, 3, 4$. Since $d(v_1) = 4$ and v_1 is (u_1, u_4) -behaved, by Lemma 3.12, $d(v_3) \geq 4$. Similarly, $d(v_2) = 4$ and v_2 is (u_1, u_2) -behaved implies that $d(v_4) \geq 4$. Thus, each f_i is a $(4, 4^+, 4^+, 4^+)$ -face for $i = 2, 3, 4$.

By Lemma 3.6(2), $d(v_3) \geq 4$ and by Lemma 3.12 $d(v_4) \geq 4$. Moreover, since $d(v_2) = 3$ and v is a poor 4-vertex and (v_1, v_3) -behaved, by Lemma 3.12 either $d(v_4) = 4$ and v_4 is not (u_3, u_4) -behaved or $d(v_4) \geq 5$. It follows that none of f_3 and f_4 is a $(3, 4, 4, 4)$ -face. We suppose that f_2 is a $(3, 4, 4, 4)$ -face and will show that $\mu^*(v) \geq 0$.

Since v is (v_2, v_4) -behaved, and $d(v_1) = d(v) = d(v_3) = 4$, by Lemma 3.12, v_1 is not (u_1, u_4) -behaved or v_3 is not (u_2, u_3) -behaved. By symmetry, we assume that v_3 is not (u_2, u_3) -behaved. This means that each of $\{u_2, u_3\}$ is incident to a triangle. So f_3 is a $(4, 4^+, 4^+, 4^+)$ -face that is adjacent to a triangle. So by Lemma 4.1(5) v gives at most $\frac{1}{4}$ to f_3 . As $d(u_2) = 4$ and u_2 is incident to a triangle, by Lemma 4.1(1) u_2 gives at least $\frac{3}{4}$ to f_2 and v_3 has at most three incident 4-faces. By Claim 4.1(3) v_3 gives at least $\frac{2}{3}$ to f_2 . So by (R1.1.2), v gives at most $2 - \frac{3}{4} - \frac{2}{3} = \frac{7}{12}$ to f_2 . Note that v gives at most $\frac{2}{3}$ to f_1 and $\frac{1}{2}$ to f_4 by Lemma 4.1(5). Thus $\mu^*(v) \geq 2 - \frac{2}{3} - \frac{7}{12} - \frac{1}{4} - \frac{1}{2} = 0$. This proves our claim.

Now we are ready to complete our proof. By Lemma 4.1, v gives at most $\frac{2}{3}$ to f_1 and $\frac{1}{2}$ to each of f_2 and f_3 . In order to show that $\mu^*(v) \geq 0$, we just need to show that v gives at most $\frac{1}{3}$ to f_4 .

We may assume that $d(v_4) \geq 5$. Note that $d(v_2) = 3$, or if $d(u_1) = 3$, then u_1 is not incident with a triangle by Proposition 3.1(c) and hence v_2 is (u_1, u_2) -behaved. It follows by Lemma 3.12 that $d(v_4) = 4$ and v_4 is not (u_3, u_4) -behaved or $d(v_4) \geq 5$. But in the former case, that means both u_3 and u_4 are incident to triangles. By Lemma 4.1(5), v gives at most $\frac{1}{4}$ to f_4 . Therefore, we may assume the latter is true, that is, $d(v_4) \geq 5$.

Assume first that v_1 is a poor 4-vertex. Then the four 4-faces incident to v_1 are f_1, f_4, f_5 and f_6 , where $f_5 = v_1 v'_1 u'_1 u_1$ and $f_6 = v_1 v'_1 u'_4 u_4$. As $d(u_1) = 3$ or $d(u_1) = 4$ and u_1 is (u'_1, v_2) -behaved, and v_1 is (v, v'_1) -behaved, by Lemma 3.12 $d(u_4) = 4$ and u_4 is not (u'_4, v_4) -behaved or $d(u_4) \geq 5$. In the former case, since v_4 is incident with a triangle and $d(v_4) \geq 5$, by Lemma 4.1(4), f_4 gains at least 1 from v_4 ; If u_4 is a poor 4-vertex, By (R1.1.2), v gives at most $\frac{2-1}{3} = \frac{1}{3}$ to f_4 ; If u_4 is rich, u_4 gives at least $\frac{1}{2}$ to f_4 , thus by (R1.1.2), v gives at most $\frac{2-1-\frac{1}{2}}{2} = \frac{1}{4}$ to f_4 . In the latter case, if at least one of u_4 and v_4 is a rich 5-vertex or 6^+ -vertex, then by (R1.2.2) and Lemma 4.1(4) v gives at most $\frac{2-1-\frac{1}{2}}{2} = \frac{1}{4}$ to f_4 ; thus, we may assume that both u_4 and v_4 are poor 5-vertices, but it follows that f_4 is a special $(4, 4, 5, 5)$ -face, and by (R1.2.2) and (R1.1.2), v gives $\frac{2-\frac{3}{4}-\frac{1}{2}}{2} = \frac{1}{4}$ to f_4 .

Now we assume that v_1 is a rich 4-vertex. Then v_1 is incident to at most three 4-faces.

We first show that v_1 cannot be incident to a $(3, 3, 4, 4^+)$ -face. Suppose otherwise that v_1 is incident to such 4-face. Note that f_1 and f_4 are not $(3, 3, 4, 4^+)$ -face. Thus assume that v_1 is incident to a $(3, 3, 4, 4^+)$ -face f_5 that share an edge with f_1 or f_4 . Let $N(v_1) = \{u_1, v, u_4, v'_1\}$. If $d(u_1) = 3$, then v_1 is (u_1, u_4) -behaved and (v, v'_1) -behaved, thus by Lemma 3.11(4) f_5 cannot be a $(3, 3, 4, 4^+)$ -face, a contradiction. If $d(v_2) = 3$, then $d(u_1) = 4$, thus by Lemma 3.11(1) and (2), f_5 cannot be a $(3, 3, 4, 4^+)$ -face, a contradiction.

Thus by Lemma 4.1(2), v_1 gives at least $\frac{2}{3}$ to f_4 . Now we consider the degree of u_4 . Recall that $d(v_4) \geq 5$ and v is a poor 4-vertex. If u_4 is a 3-vertex, then by Lemma 4.1(4) or (R1.2.2), v_4 gives at least 1 to f_4 , thus by (R1.1.2), v gives at most $2 - 1 - \frac{2}{3} = \frac{1}{3}$ to f_4 . If u_4 is a rich 4-vertex or $d(u_4) \geq 5$, then by Lemma 4.1(2)(4) and (R1.2.2), v gives at most $2 - \frac{1}{2} \cdot 2 - \frac{2}{3} = \frac{1}{3}$ to f_4 . Finally let u_4 be a poor 4-vertex. If v_4 is a poor 5-vertex, then f_4 is a special $(4, 4, 4, 5)$ -face, thus by (R1.2.2), v_4 gives $\frac{3}{4}$ to f_4 ; If v_4 is not a poor 5-vertex, then Lemma 4.1(4), v_4 gives at least 1 to f_4 . Thus, by (R1.1.2) v gives at most $\max\{\frac{2-\frac{3}{4}-\frac{2}{3}}{2}, \frac{2-1-\frac{2}{3}}{2}\} = \frac{7}{24} \leq \frac{1}{3}$ to f_4 . \square

In order to prove that 5-vertices have nonnegative charges (Lemma 4.5), we first handle two special cases in Lemmas 4.3 and 4.4.

Lemma 4.3. *Suppose that v is a poor 5-vertex and $N(v)$ has no vertex incident to a triangle. If $f_i = u_i v_{i+1} v v_i$ is a $(3, 4, 5, 4)$ -face, then $\mu^*(v) \geq 0$.*

Proof. By symmetry, let $i = 1$. First we show that none of f_2 and f_5 is a special $(4, 4, 5, 5)$ - or $(4, 4, 4, 5)$ -face. Suppose otherwise that by symmetry f_2 is a special $(4, 4, 5, 5)$ - or $(4, 4, 4, 5)$ -face. By the definition of special $(4, 4, 5, 5)$ - or $(4, 4, 4, 5)$ -face, v_2 is poor and $d(v_2) = d(u_2) = 4$. By Lemma 3.12, v_3 must be incident to a triangle, a contradiction. It follows that if f_2 (or f_5) is a $(4^+, 4^+, 4^+, 5)$ -face, then v gives at most $\frac{1}{2}$ to it.

By Lemma 3.13 (2), at most two vertices in u_i with $i \in [5]$ are 3-vertices. Since $d(u_1) = 3$, at most one of u_2 and u_5 is a 3-vertex. By symmetry we consider the following two cases.

Assume first that $d(u_5) \geq 4$ and $d(u_2) \geq 4$. If $\min\{d(v_3), d(v_5)\} \geq 4$, then v gives at most $\frac{1}{2}$ to f_2 and f_5 and at most 1 to each other incident 4-face, thus $\mu^*(v) \geq 4 - \frac{1}{2} \cdot 2 - 1 \cdot 3 = 0$. So we may assume by symmetry that $d(v_5) = 3$. By Lemma 3.6(2) $d(v_4) \geq 4$. Since $d(u_1) = 3$, by Lemma 3.13(1), $d(u_4) \geq 4$. We claim that $d(u_3) \geq 4$, for otherwise, since $d(v_5) = 3$, by Lemma 3.13(3) $d(v_2) = 4$ and v_2 is not (u_1, u_2) -behaved, or $d(v_2) \geq 5$, which is contrary to our assumption that $d(v_2) = 4$ and v_2 is (u_1, u_2) -behaved (note that u_1 cannot be incident to a triangle). Since $d(v_5) = 3$, applying Lemma 3.13 (3) to u_1 , we get $d(v_3) = 4$ and v_3 is not (u_2, u_3) -behaved or $d(v_3) \geq 5$. In the former case, f_2 is a $(4, 5, 4^+, 4)$ -face with u_2 incident to a triangle and f_3 is a $(4, 5, 4^+, 4^+)$ -face with u_3 incident to a triangle, then by (R1.2.2), v gives at most $\frac{1}{2}$ to each of f_2 and f_3 , thus, $\mu^*(v) \geq 4 - 1 \cdot 3 - 2 \cdot \frac{1}{2} = 0$. Consider the latter case now. As the argument above, v gives at most $\frac{1}{2}$ to f_2 . Note that f_3 is a $(4^+, 4^+, 5^+, 5)$ -face, so if f_3 is not special $(4, 4, 5, 5)$ -face, then by (R1.2.2) v gives at most $\frac{1}{2}$ to f_3 , and it follows that $\mu^*(v) \geq 0$; thus, we may assume that f_3 is a special $(4, 4, 5, 5)$ -face. It follows that v_4 and u_3 are both poor 4-vertices. Since v_3 and v_5 are not incident to triangles, applying

Lemma 3.12 to v_4 , we have $d(u_4) \geq 5$, so f_4 is a $(3, 5, 4, 5^+)$ -face. By (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_3 and f_4 , so $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$.

Assume now by symmetry that $d(u_5) = 3$ and $d(u_2) \geq 4$. By Lemma 3.13(1), $d(v_5) \geq 4$. By Lemma 3.13 (2) $d(u_i) \geq 4$ for $i = 2, 3, 4$. Since $d(v_2) = 4$ and v_2 is (u_1, u_2) -behaved, by Lemma 3.13 (3) (with $i = 5$), we get $d(v_4) \geq 5$ or $d(v_4) = 4$ and v_4 is not (u_3, u_4) -behaved. If $d(v_3) \geq 4$, then v gives at most $\frac{3}{4}$ to each of f_3 and f_4 and $\frac{1}{2}$ to f_2 by (R1.2.2), thus $\mu^*(v) \geq 0$. So let $d(v_3) = 3$. By Lemma 3.13 (3) (with $i = 1$), we get $d(v_5) \geq 5$ (since $d(u_5) = 3$, v_5 is not (u_4, u_5) -behaved). Since $d(v_4) \geq 5$ or $d(v_4) = 4$ but both u_3 and u_4 are incident to triangles, f_4 is a $(4^+, 4^+, 5, 5^+)$ -face but not a special $(4, 4, 5, 5)$ -face and f_3 is a $(3, 4^+, 4^+, 5)$ -face but not a special $(3, 4, 4, 5)$ -face. Thus, by (R1.2.2), v gives at most $\frac{1}{2}$ to f_4 and $\frac{3}{4}$ to each f_3 and f_5 . So $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. \square

Lemma 4.4. *Suppose that v is a poor 5-vertex and $N(v)$ has no vertex incident to a triangle. If $f_i = v_i u_i v_{i+1} v$ is a special $(3, 4, 4, 5)$ -face, then $\mu^*(v) \geq 0$.*

Proof. By symmetry, let $i = 1$. By Lemma 4.3 we may assume that v is not incident to a $(3, 4, 5, 4)$ -face. Since $d(v_1) = 3$, by Lemma 3.6(2) $d(v_5) \geq 4$. Since f_1 is a special $(3, 4, 4, 5)$ -face, u_1 is not incident to a triangle, thus at most one vertex in $\{u_1, v, u_2\}$ is incident to a triangle, so by applying Lemma 3.11(1) to v_2 , $d(v_3) \geq 4$.

We may assume that f_2 is neither a special $(4, 4, 4, 5)$ -face nor a special $(4, 4, 5, 5)$ -face. Suppose otherwise, then v_2 is poor and $d(v_2) = d(u_2) = d(u_1) = 4$, and none of v_1, v, v_3 is incident to a triangle, a contradiction to Lemma 3.12. It follows that if f_2 is a $(4, 5, 4^+, 4^+)$ -face, then by (R1.2.2), v gives at most $\frac{1}{2}$ to f_2 .

We may also assume that $d(u_5) \geq 4$. Suppose otherwise that $d(u_5) = 3$. By Lemma 3.13 (1) $d(u_i) \geq 4$ for $i = 2, 3, 4$. Since f_1 is a special $(3, 4, 4, 5)$ -face, u_1 is not incident with a triangle. Applying Lemma 3.13 (3) to u_5 , $d(v_4) \geq 4$, so by (R1.2.2), v gives at most $\frac{3}{4}$ to each f_3 and f_4 . Note that v gives at most $1/2$ to f_2 , as it is a $(4, 5, 4^+, 4^+)$ -face. But now $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$.

Now we consider the following four cases depending on the degree of u_3 and u_4 .

Let $d(u_3) = d(u_4) = 3$. By Lemma 3.7 $d(v_4) \geq 4$. By Lemma 3.13(3) (with $i = 4$) $d(v_3) \geq 5$ and (with $i = 3$) $d(v_5) \geq 5$. It follows that for $i \in \{2, 3, 4, 5\}$, f_i is a $(3^+, 4^+, 5, 5^+)$ -face, so by (R1.2.2), v gives at most $\frac{3}{4}$ to f_i . Thus, $\mu^*(v) \geq 4 - 1 - 4 \cdot \frac{3}{4} = 0$.

Let $d(u_3) = 3$ and $d(u_4) \geq 4$. Since $d(v_2) = 4$ and v_2 is (u_1, u_2) -behaved, by Lemma 3.13(3) (with $i = 3$), either $d(v_5) = 4$ and v_5 is not (u_4, u_5) -behaved or $d(v_5) \geq 5$, then f_4 and f_5 are $(3, 4^+, 4^+, 5)$ -faces but not special $(3, 4, 4, 5)$ -faces, so by (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_4 and f_5 . If $d(u_2) \geq 4$, then v gives at most $\frac{1}{2}$ to f_2 which is a $(4, 5, 4^+, 4^+)$ -face, so $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. Thus, we assume that $d(u_2) = 3$. As f_2 cannot be a $(3, 4, 5, 4)$ -face, $d(v_3) \geq 5$, so f_2 and f_3 are $(3, 4^+, 5, 5^+)$ -faces, then by (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_3 and f_2 . We conclude that $\mu^*(v) \geq 4 - 1 - 4 \cdot \frac{3}{4} = 0$.

Let $d(u_3) \geq 4$ and $d(u_4) = 3$. By Lemma 3.13(3) (with $i = 4$), $d(v_3) = 4$ and v_3 is not (u_2, u_3) -behaved or $d(v_3) \geq 5$. In the former case, f_2 is a $(4, 5, 4, 4^+)$ -face with u_2 incident to a triangle and f_3 is a $(3^+, 4^+, 4, 5)$ -face with u_3 incident to a triangle; In the latter case, each of f_2 and f_3 is a $(3^+, 4^+, 5, 5^+)$ -face; so by (R1.2.2) v gives at most $\frac{3}{4}$ to each of f_2 and f_3 . If $d(u_2) = 3$, then by Lemma 3.13 (3) (with $i = 2$), $d(v_4) \geq 5$, thus f_2 is a $(3, 4, 5, 5^+)$ -face, f_3 is a $(4^+, 5^+, 5, 5^+)$ -face and f_4 is a $(3, 4^+, 5, 5^+)$ -face, so by (R1.2.2), v gives at most $\frac{1}{2}$ to f_3 and at most $\frac{3}{4}$ to each f_2 and f_4 , therefore, $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. Now we assume that $d(u_2) \geq 4$. If $d(v_5) = 4$, then at most one of $\{u_5, v, u_4\}$ is incident with a triangle, so by applying Lemma 3.11 (1) to v_5 , we have $d(v_4) \geq 4$; as v is not incident to $(3, 4, 5, 4)$ -faces, we further conclude that $d(v_4) \geq 5$. Now, f_2 is a $(4, 5, 4^+, 4^+)$ -face, f_3 is a $(4^+, 4^+, 5, 5^+)$ -face, and f_4 is a $(3, 4, 5, 5^+)$ -face, so by (R1.2.2), v gives at most $\frac{1}{2}$ to f_2 and $\frac{3}{4}$ to each of f_3 and f_4 . It follows that $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. If $d(v_5) \geq 5$, then by (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_4 and f_5 , and gives at most $\frac{1}{2}$ to f_2 which is a $(4, 5, 4^+, 4^+)$ -face. We conclude that $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$.

We are left to consider the case that $d(u_3) \geq 4$ and $d(u_4) \geq 4$.

Assume first that $d(u_2) \geq 4$. Note that v gives $\frac{1}{2}$ to f_2 which is a $(4, 5, 4^+, 4^+)$ -face. If $d(v_4) \geq 4$, then each of f_3 and f_4 is a $(4^+, 4^+, 4^+, 5)$ -face, so by (R1.2.2), v gives $\frac{3}{4}$ to each of f_3 and f_4 , it follows that $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. So let $d(v_4) = 3$. Then both f_4 and f_5 are $(3, 4^+, 4^+, 5)$ -faces. If $d(v_5) = 4$ and v_5 is (u_4, u_5) -behaved, then at most one of $\{u_4, u_5, v\}$ is incident with a triangle, and $d(v_1) = d(v_4) = 3$, a contradiction to Lemma 3.11(1). This means either $d(v_5) = 4$ and both u_4 and u_5 are incident to triangles or $d(v_5) \geq 5$. Thus, none of f_4 and f_5 is a special $(3, 4, 4, 5)$ -face. By (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_4 and f_5 . Thus, we also have $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$.

Thus, we may assume that $d(u_2) = 3$. By Lemma 3.13 (3) (with $i = 2$), either $d(v_4) = 4$ and v_4 is not (u_3, u_4) -behaved or $d(v_4) \geq 5$. In the former case, both f_3 and f_4 are $(4, 4^+, 4^+, 5)$ -faces, but none of them is a special $(4, 4, 4, 5)$ - or $(4, 4, 5, 5)$ -face, since u_3 and u_4 are incident with triangles; by (R1.2.2), v give at most $\frac{1}{2}$ to f_3 and f_4 , thus, $\mu^*(v) \geq 4 - 3 - 2 \cdot \frac{1}{2} = 0$. So consider the latter case that $d(v_4) \geq 5$. We claim that f_3 is not a special $(4, 4, 5, 5)$ -face, for otherwise, $d(v_3) = d(u_3) = 4$ and v_3 is poor, but none of v and v_4 is incident to triangles, and $d(u_2) = 3$, a contradiction to Lemma 3.12. It follows by (R1.2.2) that v gives at most $\frac{1}{2}$ to f_3 . If f_4 is not a special $(4, 4, 5, 5)$ -face, then by (R1.2.2), v gives at most $\frac{1}{2}$ to f_4 , which implies that $\mu^*(v) \geq 4 - 3 - 2 \cdot \frac{1}{2} = 0$. Thus, we assume that f_4 is a special $(4, 4, 5, 5)$ -face. It follows that $d(u_4) = d(v_5) = 4$ and v_5 is poor. By Lemma 3.12, $d(u_5) \geq 5$. It follows that f_5 is a $(3, 5, 4, 5^+)$ -face. By (R1.2.2), v gives at most $\frac{3}{4}$ to each of f_5 and f_4 . Therefore, $\mu^*(v) \geq 4 - 2 \cdot 1 - 2 \cdot \frac{3}{4} - \frac{1}{2} = 0$. \square

Lemma 4.5. *Each 5-vertex $v \in \text{int}(C_0)$ has nonnegative final charge.*

Proof. If v is rich, then by (R1.2.1), v gives at most $2 + \max\{1 \cdot 2, 3 \cdot \frac{1}{2}\} = 4$ to incident triangles and pendant 3-faces and incident 4-faces, thus its final charge must be nonnegative. Thus, we may assume that v is poor.

We may further assume that some vertex in $N(v)$ is incident to a triangle. Suppose otherwise. By Lemmas 4.3 and 4.4, we may assume that v is not incident to a $(3, 4, 5, 4)$ -face or a special $(3, 4, 4, 5)$ -face. If v is not incident to a $(3, 3, 5, 4^+)$ -face, then by (R1.2.2), $\mu^*(v) \geq 4 - 5 \cdot \frac{3}{4} > 0$, so by symmetry, we assume that $f_1 = u_1 v_1 v v_2$ is a $(3, 3, 5, 4^+)$ -face. By Lemma 3.6(2), $d(v_5) \geq 4$. For $i \in \{2, 3, 4, 5\}$, $d(u_i) \geq 4$ by Lemma 3.13 (1), then f_i cannot be a $(3, 3, 5, 4^+)$ -face, so by (R1.2.2), v gives at most $3/4$ to f_i . It follows that $\mu^*(v) \geq 4 - 1 - \frac{3}{4} \times 4 = 0$.

Now we consider the following two cases.

Case 1. $N(v)$ has at least two vertices incident to triangles.

If v_i and v_{i+1} for some $i \in [5]$ are incident to triangles, then f_i is rich and by (R1.2.2) v gives 0 to f_i and at most 1 to each other 4-face. Thus, $\mu^*(v) \geq 4 - 4 = 0$. We assume, without loss of generality, that v_1 and v_3 are incident with triangles. If $d(v_2) \geq 4$, then by (R1.2.2), v gives at most $\frac{1}{2}$ to each of f_1 and f_2 , and gives at most 1 to each of f_3, f_4 and f_5 , so $\mu^*(v) \geq 4 - 3 \cdot 1 - 2 \cdot \frac{1}{2} = 0$. Thus, we may assume that $d(v_2) = 3$. If $\min\{d(u_1), d(u_2)\} \geq 4$, then each of f_1 and f_2 is a $(3, 4^+, 5, 5^+)$ -face but not a special $(3, 4, 5, 5)$ -face, so by (R1.2.2), v gives at most $\frac{1}{2}$ to each of f_1 and f_2 , therefore, $\mu^*(v) \geq 4 - 3 \cdot 1 - 2 \cdot \frac{1}{2} = 0$. Thus, by symmetry, assume that $d(u_1) = 3$. By Lemma 3.7 $d(u_2) \geq 4$. Note that v gives at most $\frac{1}{2}$ to f_2 . If one of f_3 and f_5 , say f_3 , is not $(3, 3, 5, 5^+)$ -face, then f_3 is a $(3, 4, 5, 5^+)$ -face, so by (R1.2.2), v gives $\frac{1}{2}$ to f_3 , therefore, $\mu^*(v) \geq 0$. Thus, we may assume that f_3 and f_5 are $(3, 3, 5, 5^+)$ -faces. It follows that $d(v_4) = d(v_5) = 3$. By Lemma 3.6(2), each of u_4 and v is incident with a triangle, a contradiction.

Case 2. $N(v)$ has exactly one vertex incident to a triangle.

We assume, without loss of generality, that v_1 is incident with a triangle. If neither f_1 nor f_5 is a $(3, 3, 5, 5^+)$ -face, then by (R1.2.2), v gives at most $\frac{1}{2}$ to each of them. This implies that $\mu^*(v) \geq 4 - 3 \cdot 1 - 2 \cdot \frac{1}{2} = 0$. Thus, by symmetry we may assume that f_5 is a $(3, 3, 5, 5^+)$ -face. It follows that $d(u_5) = d(v_5) = 3$. By Lemma 3.13(1), $d(u_i) \geq 4$ for $i \in [4]$. By Lemma 3.6(2), $d(v_4) \geq 4$. By (R1.2.2) v gives at most $\frac{1}{2}$ to f_1 .

We may assume that $d(v_4) = 4$, for otherwise, both f_3 and f_4 are $(3^+, 4^+, 5^+, 5)$ -faces, thus by (R1.2.2), v gives at most $\frac{3}{4}$ to each of them, so $\mu^*(v) \geq 4 - \frac{1}{2} - 2 \cdot \frac{3}{4} - 2 \cdot 1 = 0$. By applying Lemma 3.11 (1) on 4-vertex v_4 , we get either both u_3 and u_4 are incident to triangles or $d(v_3) \geq 4$. In the former case, none of f_3 and f_4 is a special $(3, 4, 4, 5)$ -face, thus by (R1.2.2), v gives at most $\frac{3}{4}$ to each of them, so $\mu^*(v) \geq 4 - \frac{1}{2} - 2 \cdot \frac{3}{4} - 2 \cdot 1 = 0$. Consider the latter case which $d(v_3) \geq 4$. If f_3 is neither a special $(4, 4, 4, 5)$ -face nor a special $(4, 4, 5, 5)$ -face, then by (R1.2.2), v gives at most $\frac{1}{2}$ to f_3 , so $\mu^*(v) \geq 4 - 3 \cdot 1 - 2 \cdot \frac{1}{2} = 0$. Thus, we may assume that f_3 is a special $(4, 4, 4, 5)$ -face or a special $(4, 4, 5, 5)$ -face. By (R1.2.2), v gives at most $\frac{3}{4}$ to f_3 . By the definition of special $(4, 4, 4, 5)$ -face or $(4, 4, 5, 5)$ -face, v_4 is poor and $d(v_4) = d(u_3) = 4$. Note that no vertex in $\{v_3, v, v_5\}$ is incident to a triangle. By Lemma 3.12, $d(u_4) \geq 5$. So f_4 is a $(3, 5, 4, 5^+)$ -face and by (R1.2.2) v gives at most $\frac{3}{4}$ to f_4 . Thus, $\mu^*(v) \geq 4 - (2 \cdot 1 + 2 \cdot \frac{3}{4} + \frac{1}{2}) = 0$. \square

Now we consider the case $v \in C_0$.

Lemma 4.6. *Each $v \in C_0$ has nonnegative final charge.*

Proof. We consider the following cases according to the degree of v . For $l = 3, 4$, by Lemma 3.4 each l -face f in G satisfies that $|b(f) \cap C_0| \leq 2$ and furthermore, when $|b(f) \cap C_0| = 2$, f and C_0 share a common edge.

- (1) $d(v) = 2$. By (R3), $\mu^*(v) = 2 \times 2 - 6 + 2 = 0$.
- (2) $d(v) = 3$. Then v could be incident with at most one triangle from F_3'' or has at most one pendant 3-face from F_3' . By (R2) and (R3), $\mu^*(v) \geq 2 \times 3 - 6 - \frac{3}{2} + \frac{3}{2} = 0$.
- (3) $d(v) = 4$. Assume first that v is incident with a 3-face f . If $f \in F_3'$, then by (R2) and (R3), $\mu^*(v) = 2 - 3 + 1 = 0$. If $f \in F_3''$, then it could be incident to at most one 4-face from F_4'' or adjacent to at most one pendent 3-face from F_3 . By (R2) and (R3), $\mu^*(v) \geq 2 - \frac{3}{2} - 1 + 1 = \frac{1}{2} > 0$. Thus, we may assume v is not incident to a 3-face. By Lemma 3.6 (1), v is not incident face from F_4' . Thus, we assume that v is incident with $k \leq 2$ 4-faces from F_4'' . Then v is adjacent to at most $2 - k$ pendent 3-faces from F_3 . By (R2) and (R3), $\mu^*(v) \geq 2 - k - \frac{1}{2}(2 - k) \geq 0$.
- (4) $d(v) = k \geq 5$. If v is not incident with any 3-face, then by Lemma 3.6, v is not incident face from F_4' , so by (R2), $\mu^*(v) \geq 2k - 6 - 1 \cdot (k - 2) \geq 1 > 0$. Thus, we first assume that v is incident with a face from F_3' . Let s be the number of 4-faces in F_4' incident with v . If $s = 0$, then by (R2), $\mu^*(v) \geq 2k - 6 - (k - 4) - 3 \geq 0$; and if $s \geq 1$, then $s \leq k - 5$. By (R2), $\mu^*(v) \geq 2k - 6 - 3 - \frac{3}{2}s - (k - s - 4) = k - \frac{1}{2}s - 5 \geq \frac{1}{2}$. Next, we assume that v is incident with a face from F_3'' . If $s = 0$, then by (R2), $\mu^*(v) \geq 2k - 6 - (k - 3) - \frac{3}{2} \geq \frac{1}{2}$; if $s \geq 1$, then $s \leq k - 4$. By (R2), $\mu^*(v) \geq 2k - 6 - \frac{3}{2}s - \frac{3}{2} - (k - s - 3) = k - \frac{1}{2}s - \frac{9}{2} \geq \frac{1}{2}s - \frac{1}{2} \geq 0$. \square

Then we consider faces. As G contains no 5-faces, and 6^+ -faces other than C_0 are not involved in the discharging procedure, we only need to show that C_0 , and 3-faces and 4-faces other than C_0 have nonnegative charges.

Lemma 4.7. *Each 3-face $f \neq C_0$ has nonnegative final charge.*

Proof. Note that f has initial charge $3 - 6 = -3$. By Lemma 3.4 $|b(f) \cap C_0| \leq 2$. If $|b(f) \cap C_0| = 1$, then by (R2), $\mu^*(f) \geq -3 + 3 = 0$; if $|b(f) \cap C_0| = 2$, then by (R2), $\mu^*(f) \geq -3 + \frac{3}{2} \times 2 = 0$. Thus, we may assume that $b(f) \cap C_0 = \emptyset$. Let $f = uvw$ with corresponding degrees (d_1, d_2, d_3) . Let x' be the pendant neighbor of x on a 3-face f . By Lemma 3.8 (1), we only need to check the following cases:

- (1) f is a $(3, 3, 5^+)$ -face. By Lemma 3.8 (2), u' and v' are either on C_0 or have degree at least 4. By (R1.1.1) and (R1.2.1), f receives $\frac{1}{2}$ from each of u' and v' . By (R1.2.1) and (R1.3), f receives 2 from w . Thus, $\mu^*(f) = -3 + \frac{1}{2} \times 2 + 2 = 0$.

- (2) f is a $(3, 4, 4)$ -face. By Lemma 3.8 (3), Then u' is either on C_0 or has degree at least 4. By (R1.1.1) and (R1.2.1), f receives $\frac{1}{2}$ from u' . By (R1.1.1), f receives $\frac{5}{4}$ from each of v and w . Thus, $\mu^*(f) = -3 + \frac{5}{4} \times 2 + \frac{1}{2} = 0$.
- (3) f is a $(3, 4, 5)$ -face. By (R1.1.1) and (R1.2.1), f receives 1 from v and 2 from w . Thus, $\mu^*(f) = -3 + 1 + 2 = 0$.
- (4) f is a $(3, 5, 5)$ -face. By (R1.2.1), f receives $\frac{3}{2}$ from each of v and w . Thus, $\mu^*(f) \geq -3 + \frac{3}{2} \times 2 = 0$.
- (5) f is a $(3, 4^+, 6^+)$ -face. By (R1.1.1), (R1.2.1) and (R1.3), f receives at least 1 from v and 2 from w . Thus, $\mu^*(f) \geq -3 + 1 + 2 = 0$.
- (6) f is a $(4^+, 4^+, 4^+)$ -face. By (R1.1.1), (R1.2.1) and (R1.3), f receives at least 1 from each of u, v and w . Thus, $\mu^*(f) \geq -3 + 1 \times 3 = 0$.

□

Lemma 4.8. *Each 4-face $f \neq C_0$ has nonnegative final charge.*

Proof. Let $f = uvwx$ with corresponding degrees (d_1, d_2, d_3, d_4) . Note that f has initial charge $4 - 6 = -2$. By Lemma 3.4 $|b(f) \cap C_0| \leq 2$. If $|b(f) \cap C_0| = 1$, say $u \in b(f) \cap C_0$, then by (R2), u gives $\frac{3}{2}$ to f ; By Lemma 3.6 each of u and w is incident to a triangle, so $d(w) \geq 4$ and by (R1.1.1), (R1.2.1) and (R1.3), w gives at least $\min\{\frac{3}{4}, 1, \frac{6-2}{3}\} = \frac{3}{4}$ to f ; So $\mu^*(f) \geq -2 + \frac{3}{2} + \frac{3}{4} > 0$. If $|b(f) \cap C_0| = 2$, then by (R2), $\mu^*(f) \geq -2 + 1 \times 2 = 0$. So we now assume that $b(f) \cap C_0 = \emptyset$. If some vertex on $b(f)$ is poor 4-vertex, then by (R1.1.2), the poor 4-vertex will give enough charges to f to make its final charge to be 0. So we assume that each 4-vertex on $b(f)$ is rich. By Lemma 3.6 (2), we only need to consider the following 4-faces.

- (1) f is a $(3, 3, 4^+, 4^+)$ -face. By (R1.1.1), (R1.2) and (R1.3), each 4^+ -vertex gives at least 1 to f . Thus, $\mu^*(f) \geq -2 + 1 \times 2 = 0$.
- (2) f is a $(3, 4^+, 3, 4^+)$ -face. Then by Lemma 3.6(2), and both v and x are incident to a triangle. By Lemma 4.1 (1), f receives at least 1 from each of v and x , so $\mu^*(f) \geq -2 + 1 \times 2 = 0$.
- (3) f is a $(3, 4, 4, 4)$ -face. If w is not incident to a triangle, then by Lemma 4.1(3), each of the rich 4-vertices gives at least $\frac{2}{3}$ to f . Thus, $\mu^*(f) \geq -2 + 3 \cdot \frac{2}{3} = 0$. Let w be incident to a triangle f_1 . Note that w is rich and none of v and x is poor. If f_1 is a $(3, 4, 4)$ -face, then by Lemma 3.9 (1), each of v and x is incident with a triangle. In this case, by Lemma 4.1(1), each of v, w and x gives at least $\frac{3}{4}$ to f . This implies that $\mu^*(f) \geq -2 + 3 \cdot \frac{3}{4} > 0$. Thus, assume that f_1 is not a $(3, 4, 4)$ -face. By (R1.1.1) and Lemma 4.1(2), w gives at least 1 to f and each of v and x gives at least $\frac{1}{2}$ to f . Thus, $\mu^*(f) \geq -2 + 1 + 2 \cdot \frac{1}{2} = 0$.
- (4) f is a $(3, 4, 4, 5^+)$ -face. First we assume that x is a 6^+ -vertex or x is a rich 5-vertex, then by Lemma 4.1 (2) and (4), f receives at least 1 from x and $\frac{1}{2}$ from each of v and w , thus $\mu^*(f) \geq -2 + 1 + \frac{1}{2} \times 2 = 0$. Now we assume that x is a poor 5-vertex. If none of the two 4-vertices is incident to a triangle, then f is a special $(3, 4, 4, 5)$ -face. By (R1.2.2) and Lemma 4.1 (2), f receives at least 1 from x and $\frac{1}{2}$ from each of v and w . If both of the two 4-vertices are incident to triangles, then by Lemma 4.1 (2) and (R1.2.2), f gets at least $\frac{3}{4}$ from each of v and w and $\frac{1}{2}$ from x . If exactly one of the two 4-vertices (say v) is incident to a triangle, then f is a weak $(3, 4, 4, 5)$ -face. By (R1.2.2) and Lemma 4.1 (2), f receives at least $\frac{3}{4}$ from each of v and x and $\frac{1}{2}$ from w . In both cases, $\mu^*(f) \geq 2 - \max\{1 + \frac{1}{2} \cdot 2, \frac{3}{4} \cdot 2 + \frac{1}{2}\} = 0$.
- (5) f is a $(3, 4, 5^+, 4)$ -face. By (R1.2.2) and Lemma 4.1 (2) and (4), f receives at least 1 from w and $\frac{1}{2}$ from each of v and x . Thus, $\mu^*(f) \geq -2 + 1 + \frac{1}{2} \times 2 = 0$.
- (6) f is a $(3, 4, 5^+, 5^+)$ -face or $(3, 5^+, 4, 5^+)$ -face. By (R1.2.2) and Lemma 4.1 (2) and (4), f receives at least $\frac{3}{4}$ from each of the two 5^+ -vertices, and $\frac{1}{2}$ from the 4-vertex. Thus, $\mu^*(f) = -2 + \frac{3}{4} \cdot 2 + \frac{1}{2} = 0$.
- (7) f is a $(3, 5^+, 5^+, 5^+)$ -face. If at least one vertex is a rich 5-vertex or a 6^+ -vertex, then by Lemma 4.1(4) and (R1.2.2), f gets at least 1 from the vertex and at least $\frac{1}{2}$ from each of the other 5-vertices. It follows that $\mu^*(f) \geq -2 + 1 + 2 \cdot \frac{1}{2} = 0$. Thus, we may assume that all are poor 5-vertices. In this

case, by (R1.2.2), f is special $(3, 5, 5, 5)$ -face. Thus f receives $\frac{3}{4}$ from each of the 5-vertices. Thus, $\mu^*(f) \geq -2 + 3 \cdot \frac{3}{4} > 0$.

- (8) f is a $(4^+, 4^+, 4^+, 4^+)$ -face. If f is rich, then f contains at least two rich 5^+ vertices or 6^+ -vertices. By Lemma 4.1(4), $\mu^*(f) \geq -2 + 1 \cdot 2 = 0$. Thus, we may assume that f is not rich. By Lemma 4.1(2) each rich 4-vertex gives at least $\frac{1}{2}$ to f . By Lemma 4.1(4) and (R1.2.2) each 5^+ -vertex gives at least $\frac{1}{2}$ to f . Note that each 4^+ -vertex on f is not poor 4-vertex. Thus, f receives at least $\frac{1}{2}$ from each vertex on $b(f)$. So $\mu^*(f) \geq -2 + 4 \cdot \frac{1}{2} = 0$.

□

Now we consider the outer-face C_0 . Let t_i be the number of i -vertices on C_0 , then $d(C_0) \geq t_2 + t_3 + t_4$. Note that $d(C_0) \in \{3, 7\}$. By (R3),

$$\begin{aligned} \mu^*(C_0) &= d(C_0) + 6 - 2t_2 - \frac{3}{2}t_3 - t_4 \geq d(C_0) + 6 - \frac{3}{2}(t_2 + t_3 + t_4) - \frac{t_2}{2} \\ &\geq d(C_0) + 6 - \frac{3}{2}d(C_0) - \frac{t_2}{2} = 6 - \frac{d(C_0)}{2} - \frac{t_2}{2}. \end{aligned}$$

If $d(C_0) = 3$ or $t_2 \leq 5$, then $\mu^*(C_0) \geq 0$. Thus, we may assume that $d(C_0) = 7$ and $(t_2, t_3, t_4) \in \{(6, 1, 0), (7, 0, 0)\}$. If $t_2 = 7$, then $G = C_0$ and it is trivially superextendable. If $t_2 = 6$ and $t_3 = 1$, then by (R3), C_0 gains 1 from the adjacent face which has degree more than 7. Thus, $\mu^*(C_0) \geq \frac{1}{2} > 0$.

We have shown that all vertices and faces have non-negative final charges. Furthermore, the outer-face has positive charges, except when $d(C_0) = 7$ and $t_2 = 5$ and $t_3 = 2$ (the two 3-vertices must be adjacent and has a common neighbor not on C_0) in which there must be a face other than C_0 having degree more than 7. Thus the face has positive final charge. Therefore, $\sum_{x \in V(G) \cup F(G)} \mu^*(x) > 0$, a contradiction.

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